Mixed Base Counting: Math Created to Solve a Programming Problem

by Laura Shears

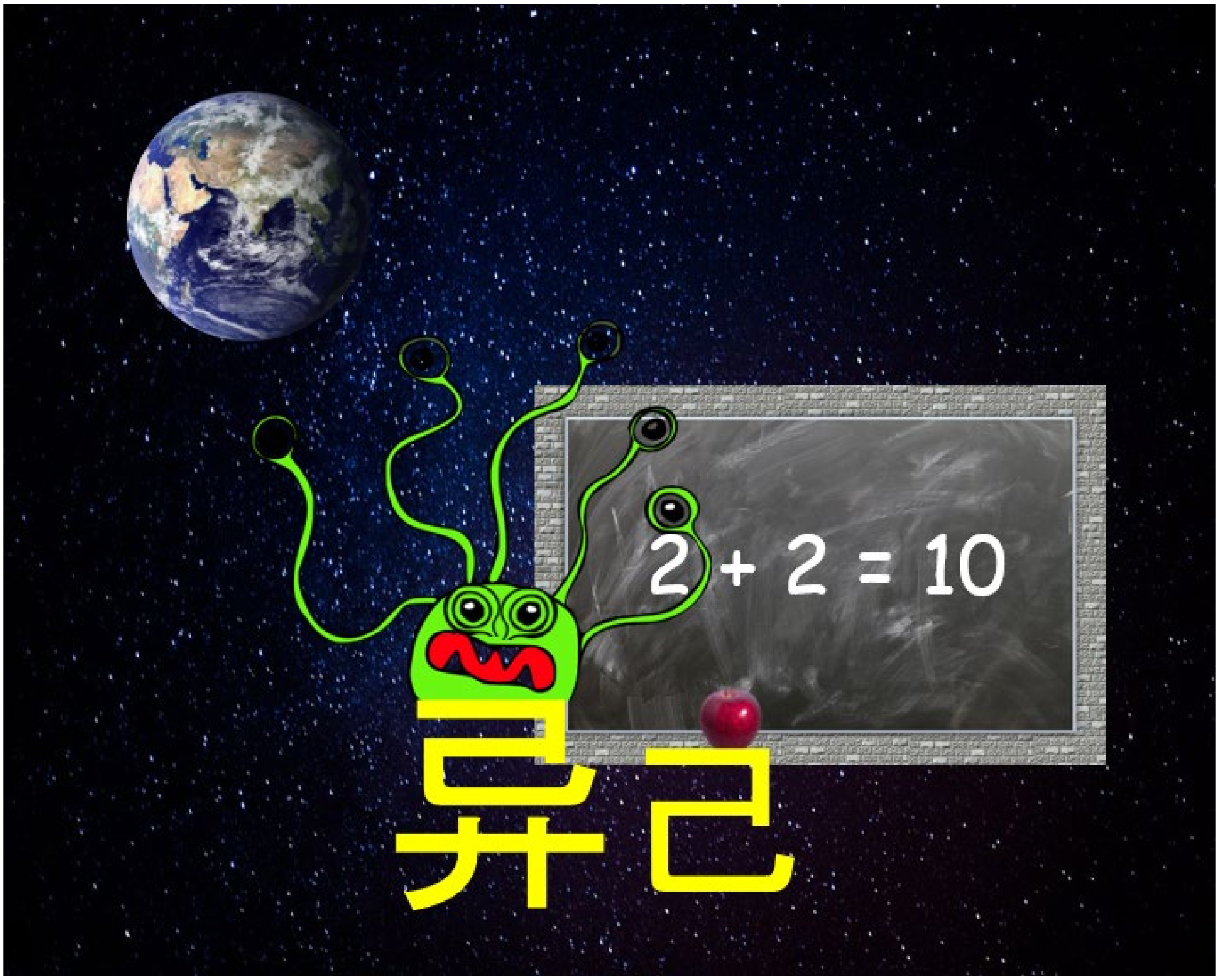


Image by Steven Berg with parts created by others as mentioned below.

Alien Math (2019)

This art piece was designed by Steven Berg for his 2020 LAND Presentation “Dancing on the Precipice.” It was inspired by my “Mixed Base Arithmetic” proposal which will also be presented at the conference.

The night sky was designed by Nikiko. The blackboard is by geralt and OpenClipart-Vectors created the alien. The apply was by Capri23auto. All of the images were released on Pixabay.

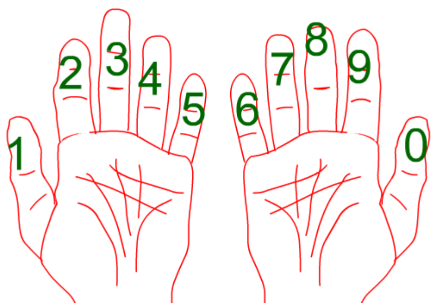
The Chinese characters spell out alien.

Created February 5, 2020. Updated February 8, 2020

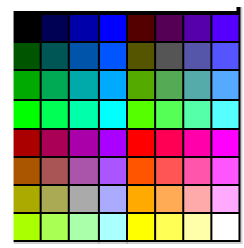
This is a work in progress. Check back before each use for updates. You might want to use the most recent version.

Section R: Alien Arithmetic Review

When counting in a different base counting system, you probably will start thinking about how are own system works. Most of us have ten fingers, also known as digits and the system we use also has ten numerical digits as in the image below. An alien that only has 4 fingers might use a base 4 counting system.



There are many uses for counting in different base counting systems. We use base 16 to control colors on webpages and in other computer programs. Base 2 is used to control computers since every switch in a computer has 2 choices: on or off. If you are interested in advanced math, look up the Cantor set and you are likely to find a proof about the number of elements in it that uses base 3. I once created a color selector for some of my programs that uses base 4 because I offered 4 shades each of the three primary colors of light: red, green, and blue. Different base arithmetic is useful almost anytime where there is a need to mix things with a number of choices other than ten.



Section R Problems. Alien Arithmetic in Base 4

1.) Show you can count by filling in the blanks in the counting chart below.

Fill in the missing numbers for the base 4 counting chart below.
After filling in a blank, use your tab key to get to the next blank.

1	2	3	10
11		13	20
21	22	23	30
31	32	33	100
101	102	103	110
111	112	113	120
121	122	123	130
131	132	133	
201	202	203	210
211	212	213	
221	222	223	230
231	232	233	300
301	302		310
311	312	313	320
321	322	323	330
331	332	333	1000

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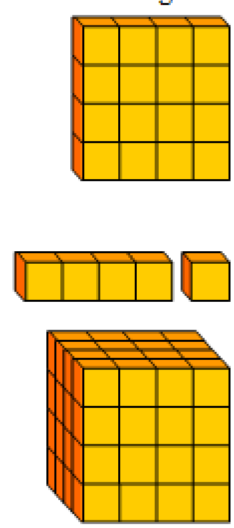
Base 4 Counting Blocks



Preform the arithmetic in the following problems using base 4. You can use the chart above or just use borrowing and carrying like you do for base ten. Think about counting blocks for multiplication and division.

2.)

$$\begin{array}{r} 330 \\ +\quad 323 \\ \hline \end{array}$$

3.)

$$\begin{array}{r} 221 \\ -\quad 3 \\ \hline \end{array}$$

4.) $20 \times 13 =$ ______

5.) $32 \div 3 =$ ______

Section 1: The Problem that Inspired Mixed Base Counting: Factoring

Example: Let's look at the prime factors of 75. Answer: $5^2 \times 3^1$

Now how about writing all of the composite and prime factors of 75. Answer: 1, 3, 5, 15, 25, 75

Now let's teach a computer to write all of the composite (including prime) factors of any positive integer up to a number that the computer can handle. That's where the trouble begins.

Of course, you could have a computer just try every integer up to the square root of the number. But what if you want to do it the way a human does it with no dead ends? Then you need to put together all of the combinations for the different number of the prime factors.

If we put all of the factors in a hat and draw one out, how many choices are there for the number of factors of 5? Answer: 3 choices. You can have 0, 1, or 2 twos.

How many choices for the number of factors of 3? Answer: 2 choices. You can have 0 or 1 factor of 3.

Normally in a computer program if you want to put together different numbers of choices for different items, you use a nested for loop.

For $i = 0$, where $i < 3$ (# of choices for the factor 5), add 1 to i after each iteration and do the following.

For $j = 0$, where $j < 2$ (# of choices for the factor 3), add 1 to j after each iteration and do the following.

Display: $5^i \times 3^j$.

This routine would result in the answer: 1, 3, 5, 15, 25, 75. Then you could run a number sort routine if needed to get the numbers in order. Normally an experienced programmer can program the computer to do nested loops quite easily. The problem here is that we don't know ahead of time how many nested loops are needed. If we are working with a number that has 5 distinct prime factors then we would need 5 nested loops.

Mixed Based Arithmetic (MBA) to the Rescue: Once the programmer writes a routine that gets all of the prime factors of a number along with their exponents, then they can use all of the exponents to determine the number of choices for each prime number. Each number of choices will be a different base in our counting system.

In our example of working with the number 75, we can expect to have 3×2 factors since there were 3 choices for the number of 5's and 2 choices for the number of 3's. If we use bases [3, 2] in that order, then we can count to $6_{\text{base ten}}$ in our mixed base system as follows: 00, 01, 10, 11, 20, 21. Next we can use our mixed base numbers as exponents. This gets us $5^0 \times 3^0 = 1$, $5^0 \times 3^1 = 3$, $5^1 \times 3^0 = 5$, $5^1 \times 3^1 = 15$, $5^2 \times 3^0 = 25$, $5^2 \times 3^1 = 75$. Thus our routine changes from an unknown number of for loops to a single for loop:

For $i = 0$, where $i < 6$ (# of factors), add 1 to i after each iteration and do the following.

- Convert i to our mixed base number.
- Use the digits of our mixed base number as exponents on our prime factors and multiply the results.
- Display the results.

Over time, there should be more applications of MBA created. Anytime a problem that would normally be solved with nested for loops, but it is unknown how many loops are needed, MBA would be a possible solution. Once you have the conversion code in your library, you may even want to use MBA when the number of nested for loops is known if there are several loops just to clean up your code.

I will leave it to the coders among us to figure out the details of the above routines. For the rest of this text, let's focus on Mixed Base Arithmetic itself.

Section 1 Problem

1. Use mixed base counting to list all of the factors of 360 like we did for 75 above.

    a) First find the prime factors. (There are 3 distinct prime factors)
    b) Determine the number of choices for each prime factor.
    c) Set up your mixed base counting chart to go with each number of choices.
    d) Count from 0 until you are at your maximum 3 digit mixed base number.
    e) Use your mixed base numbers as the exponents on the prime factors.
    f) Simplify each number and you should have a complete list of factors.

Note, another way to represent a base bigger than ten would be to use the letter that we associate with the base in counting systems larger than that base. For example in base thirteen or above, the number digit twelve would be represented with the letter 'c'. However in base twelve itself, we would represent twelve with '10'. So counting to 10 in base twelve would look like: 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, 10. Notice that this matches how we counted to 10 in base four: 1, 2, 3, 10. To avoid confusion, I would like to start incorporating letters for the base value when noting what MB we are working in. Thus MB [10, 12, 5] would become [a, c, 5]. However as long as we know that we are using base ten to denote our MB system, either notation would be considered correct.

Section 2: Mixed Base (MB) to Base Ten (BT)

In base ten a four digit integer such as

$$\begin{aligned} 5327 &= 5\times10^3 + 3\times10^2 + 3\times10^1 + 7\times10^0 \\ &= 5\times1000 + 3\times100 + 3\times10 + 7\times1 \end{aligned}$$

In MB with bases from largest to smallest place value of [4, 9, 2, 3] we can convert the number 3512 to base ten using a similar face value times place value method. Starting with 1 in the right most column (just like in base ten), the place value of the next column is the preceding columns place value times its base value. Thus the place values for this MB example are: (4×9×2×3×1, 9×2×3×1, 2×3×1, 3×1, 1) = (216, 54, 6, 3, 1). Thus $3512_{\text{MB}[4,9,2,3]} = 3\times54 + 5\times6 + 1\times3 + 2\times1 = 197_{\text{base ten}}$

Section 2 Problems

1. Convert b35 in mixed base [12, 5, 7] to base ten. Note that for bases larger than ten, we add digits by using the alphabet starting with a, b, c, etc.

2. Make up more bases and numbers and practice converting them. Discuss your answers with other students.

Section 3: Base Ten (BT) to Mixed Base (MB)

Converting BT to MB involves using division and looking at the whole number and the remainder. This is similar to converting time. Suppose we want to convert 2,562,468,263 seconds to weeks, days, hours, minutes, and seconds. First you would divide by 60 to get the number of minutes: 42,707,804.38333… minutes = 42,707,804 minutes + 23 seconds. Next divide 42,707,804 by 60 again, to get the number of hours. 711,796.7333… hours = 711,796 hours + 44 minutes. 711,796 hours ÷ 24 hours/day = 29,658 days + 10 hours. 29,658 days ÷ 7 days/week = 4236 weeks + 6 days. Putting this all together and we get 2,562,468,263 seconds = 4236 weeks, 6 days, 10 hours, 44 minutes, plus 23 seconds. This comes out to be about 79 years and 7 months. Phew, the seconds of your life go by quickly. Make the most of them!

Example: Convert 1481 in base ten to mixed base [12, 4, 9, 2, 3] = [c, 4, 9, 2, 3].

Solution: First let's get the place value of each column in this MB system like we did in Section 2: (2592, 216, 54, 6, 3, 1). We will have a 0 in the furthest left position, since 1481 < 2592, or we can just leave it off. For each of the following translate $n \div d = a\text{R}b$ to $n \div d$ equals a with remainder b.

$$1481 \div 216 = 6\text{R}185.$$
$$185 \div 54 = 3\text{R}23.$$
$$23 \div 6 = 3\text{R}5.$$
$$5 \div 3 = 1\text{R}2.$$
$$2 \div 1 = 2\text{R}0.$$

Putting this all together and we get $1481_{\text{base ten}} = 63312_{\text{MB}[c, 4, 9, 2, 3]}$.

Section 3 Problems

Convert each of the following from base ten (aka base a) to the indicated base.

1. 348 to MB [c, 4, 9, 2, 3].
2. 1680 to MB [2, 5, a, c, 2]
3. 4200 to MB [2, a, b, c, 2]

Section 4: Project for Computer Science and Math Clubs

Have your computer science students see if they can teach a computer to do the conversions and then apply what they learn to solve programming problems such as the factoring problem I used it for or some other problem. They will need to know the number of expected outcomes and loop through them all. In each loop, they will need to convert the number to their mixed base system so they can apply it their problem

Have your math club take counting to the next level and see if they can figure out how to do arithmetic like the arithmetic done in Section R for base 4.

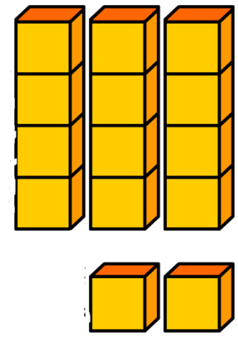
I am sure that must be more applications of this concept. Let me know what you come up with! Go to http://Mathematical.website to find contact information.

Answers to the Odd Numbered Problems

Section R

1.) Fill in the missing numbers for the base 4 counting chart below.
After filling in a blank, use your tab key to get to the next blank.

1	2	3	10
11	12	13	20
21	22	23	30
31	32	33	100
101	102	103	110
111	112	113	120
121	122	123	130
131	132	133	200
201	202	203	210
211	212	213	220
221	222	223	230
231	232	233	300
301	302	303	310
311	312	313	320
321	322	323	330
331	332	333	1000



3.) 212

5.) 10 with remainder 2

Section 1

1.) a.) $360 = 3^2 \times 2^3 \times 5^1$ b.) There are $3 \times 4 \times 2 = 24$ factors.

Other steps on the right:

$[3,4,2]$	base^{power}	product
000	$3^0 \times 2^0 \times 5^0$	1
001	$3^0 \times 2^0 \times 5^1$	5
010	$3^0 \times 2^1 \times 5^0$	2
011	$3^0 \times 2^1 \times 5^1$	10
020	$3^0 \times 2^2 \times 5^0$	4
021	$3^0 \times 2^2 \times 5^1$	20
030	$3^0 \times 2^3 \times 5^0$	8
031	$3^0 \times 2^3 \times 5^1$	40
100	$3^1 \times 2^0 \times 5^0$	3
101	$3^1 \times 2^0 \times 5^1$	15
110	$3^1 \times 2^1 \times 5^0$	6
111	$3^1 \times 2^1 \times 5^1$	30
120	$3^1 \times 2^2 \times 5^0$	12
121	$3^1 \times 2^2 \times 5^1$	60
130	$3^1 \times 2^3 \times 5^0$	24
131	$3^1 \times 2^3 \times 5^1$	120
200	$3^2 \times 2^0 \times 5^0$	9
201	$3^2 \times 2^0 \times 5^1$	45
210	$3^2 \times 2^1 \times 5^0$	18
211	$3^2 \times 2^1 \times 5^1$	90
220	$3^2 \times 2^2 \times 5^0$	36
221	$3^2 \times 2^2 \times 5^1$	180
230	$3^2 \times 2^3 \times 5^0$	72
231	$3^2 \times 2^3 \times 5^1$	360

Section 2

1.) $11 \times 35 + 3 \times 7 + 5 = 411$

Section 3

1.) 12,400

3.) 15a00